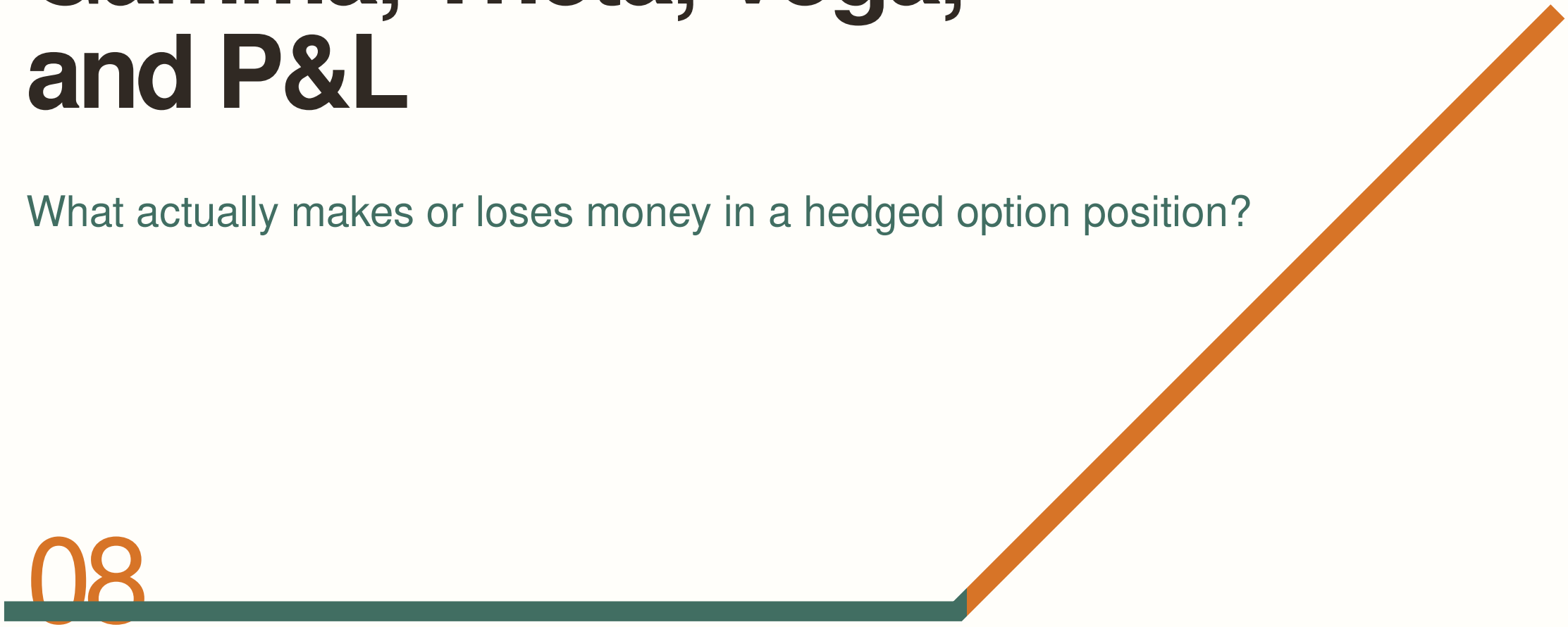
Gamma, Theta, Vega, and P&L

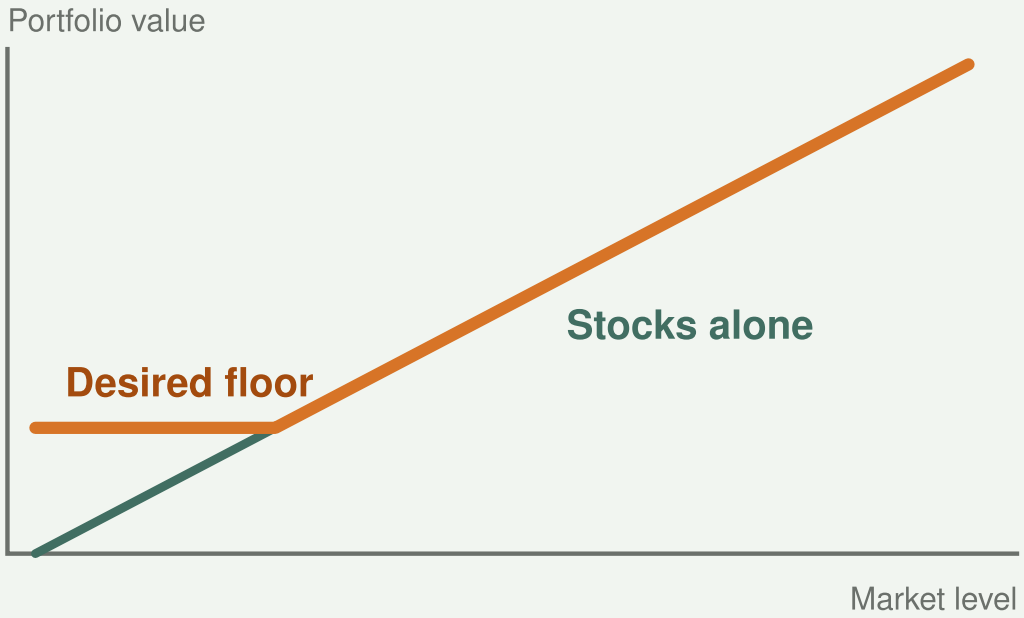
What actually makes or loses money in a hedged option position?

08



1987: insurance without a put

A fund owns stocks. It wants the upside, but it also wants a floor under its losses.



WHY NOT BUY A PUT?

Index puts were thin, short-dated, and limited in strike for large funds.

THE BUSINESS

Funds wanted protection. Banks and specialist firms sold it for fees.

THE TRADE

Build the floor by selling more index futures as the market falls.

For the fund, this is a synthetic long put. A bank selling the protection is short the put-like exposure; selling futures hedges it.

Stuff goes down, must sell more.

Wait a minute.

BLACK MONDAY · OCTOBER 19, 1987

–22.6%

Dow Jones Industrial Average. One day.



1 The market falls → 2 Portfolio insurers sell index futures → 3 Their selling adds pressure to the fall

Portfolio insurance amplified the fall. It was not the only cause.

A stock that kept landing on $10

J.D. Edwards, 2001. A fund repeatedly sold huge blocks of options at one strike.



MARCH $10 PUTS

56,128

open contracts

SIX EXPIRIES

Four

finished at the heavily traded strike

With dealers long those options, hedge trades push toward the strike.

Opposite stock trades. Why?

1987. BUILD PUT-LIKE PROTECTION.

Market falls → sell more

The trade follows the fall.

EXPIRY. HEDGE OPTIONS ALREADY OWNED.

Below strike → buy
Above strike → sell

The trade presses toward the strike.

In both stories, continuously delta hedging derivatives causes unusual movements in the underlying. Today we'll talk about how quickly we have to change our hedge, hopefully without moving the underlying.

Last time: continuously adjusted hedging

Ten calls on 100 shares each. At $S = \$100$, each call has delta 0.520, so we short about 520 shares.

START AT $100

Call delta 0.520. Short 520 shares.

STOCK RISES TO $102

Call delta 0.598. Sell about 78 more shares at $\$102$.

STOCK RETURNS TO $100

Call delta 0.520. Buy those 78 shares back at $\$100$.

That round trip collects about $\$156$ in **gross hedge-trading cash**. It is not yet the position's net profit.

Gamma measures how fast delta changes

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$$

WHAT IT MEASURES

How much option delta changes for a $\$1$ stock move, with other inputs fixed.

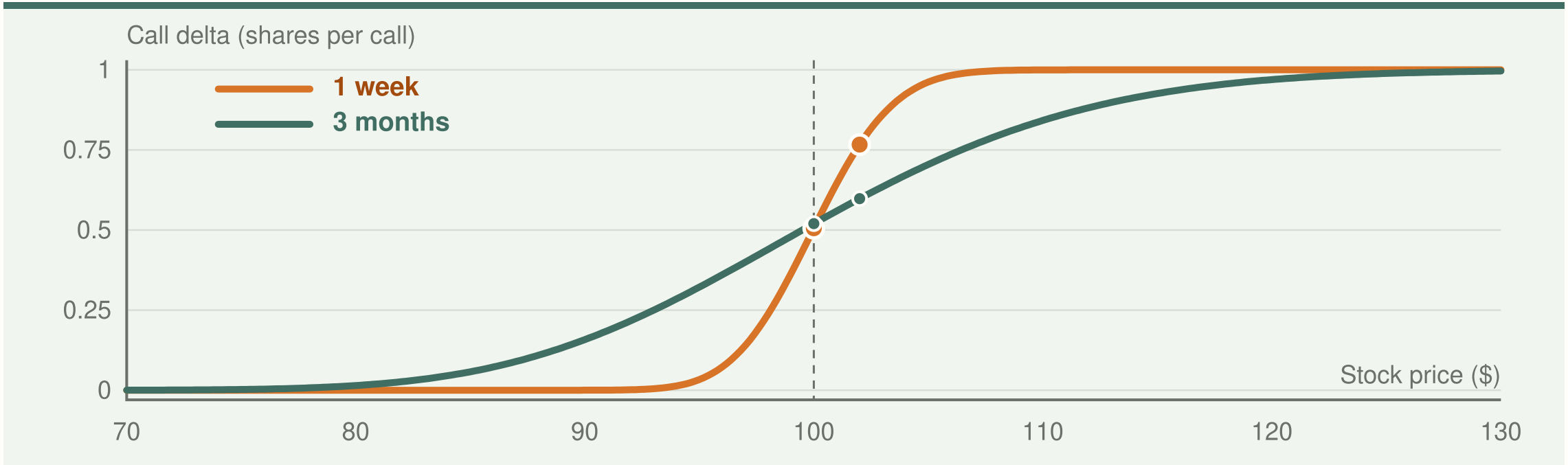
OUR CALL

From $\$100$ to $\$102$, delta rises from 0.520 to 0.598. Average gamma over that move is about 0.039 per dollar.

More gamma means a hedge goes stale faster. It also means more of those favorable buy-low, sell-high rebalances.

More gamma means more hedge trading

Same $100 strike, same 20% volatility. Compare calls with one week and three months left.



From $100 to $102, one-week delta jumps from 0.51 to 0.77. Three-month delta only moves from 0.52 to 0.60. The shorter-dated hedge goes stale faster.

Why is this not free money?

Those sell-high, buy-low stock trades are only one side of the position. You paid for the call.

WITHOUT THE CALL

Suppose you keep selling stock as it rises, hoping to buy it back. If stock keeps rising instead, the short stock loses more and more. In that case, high volatility *hurts* you.

WITH THE CALL

The call gains value as stock rises. Its non-negative payoff at expiry protects the short stock against a runaway rally. If stock reverses, you can buy back the added short at a lower price.

The call limits losses from a rally. The short stock offsets the call's directional exposure. You pay a premium for the call, and if stock barely moves, its time value decays.

What about a day where nothing happens?

Our $100 call starts with three months left, 20% volatility, and a price of about $3.988. Assume zero rates and no dividends.

TODAY		TOMORROW
$3.988	One trading day passes. Stock stays at $100.	$3.956

No stock move means no gamma gain. The long call loses about **3.2 cents per share** of time value.

Theta measures time decay

Let t be elapsed model time and T be time remaining. As a day passes, t increases while T decreases.

$$\Theta = \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial T}$$

FOR OUR CALL

$\Theta \approx -\$7.97$ per year, or about $-\$0.0316$ per trading day, per share.

INTERPRETATION

Hold stock and implied volatility fixed. A long call loses value as its remaining life shrinks.

Two effects in one hedged position

The stock hedge removes the option's immediate directional exposure. What remains over a short interval?

STOCK MOVES

Gamma gain

Curvature rewards movement in either direction.

versus

TIME PASSES

Theta decay

The long option bleeds if nothing happens.

A delta-hedged long option **loses value as time passes** and **gains from stock movement**.

What would a fair price imply?

If stock barely moves, theta wins. The hedged long option loses money.

If stock moves a lot, gamma wins. The hedged long option gains money.

At a fair Black-Scholes price, the two cancel **in expectation under the model**, before costs.

The model does not promise zero P&L on every path. Implied volatility sets the benchmark: were the stock's moves large enough to offset time decay?

Pause



Calculate gamma and theta

For the derivations, use the Black-Scholes call from Lecture 05. Set rates and dividends to zero so the gamma-theta tradeoff is visible by itself.

$$C = SN(d_1) - KN(d_2), \qquad d_1 = \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

$\Delta = C_S$
Stock sensitivity

$\Gamma = C_{SS}$
How delta changes

$\Theta = C_t = -C_T$
Calendar-time effect

Derive gamma from delta

Lecture 07 gave us $\Delta = N(d_1)$. Differentiate once more with respect to stock price.

$$\Gamma = \frac{\partial N(d_1)}{\partial S} = \phi(d_1)\frac{\partial d_1}{\partial S}.$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}.$$

$$\boxed{\Gamma = \frac{\phi(d_1)}{S\sigma\sqrt{T}} > 0}.$$

ϕ is the standard normal density. Both calls and puts have this same gamma when they share S, K, T, σ.

Derive theta from time remaining

Change time remaining, T, while holding S, K, and σ fixed. From Lecture 07, $S\phi(d_1) = K\phi(d_2)$, and $d_1 - d_2 = \sigma\sqrt{T}$.

$$\frac{\partial C}{\partial T} = S\phi(d_1)\frac{\partial d_1}{\partial T} - K\phi(d_2)\frac{\partial d_2}{\partial T}.$$

$$\frac{\partial C}{\partial T} = S\phi(d_1)\frac{\partial (d_1 - d_2)}{\partial T} = \frac{S\phi(d_1)\sigma}{2\sqrt{T}}.$$

$$\boxed{\Theta = -\frac{\partial C}{\partial T} = -\frac{S\phi(d_1)\sigma}{2\sqrt{T}} < 0}.$$

Where does hedged P&L come from?

Hold the old delta hedge over a short interval. Write the stock move as δS and elapsed time as δt.

$$\delta C \approx \Delta\,\delta S + \Theta\,\delta t + \frac{1}{2}\Gamma(\delta S)^2.$$

$$\delta(\text{short } \Delta \text{ shares}) = -\Delta\,\delta S.$$

$$\boxed{\text{Hedged P\&L} \approx \Theta\,\delta t + \frac{1}{2}\Gamma(\delta S)^2}.$$

Rebalance **after** the move. That trade resets the next interval's hedge, but does not change the P&L already earned.

Gamma pays for theta in the model

From Lecture 05, $\mathbb{E}[(\delta S)^2] \approx \sigma^2 S^2 \delta t$. Put this into the hedge P&L from the previous slide.

$$\text{Hedged P\&L} \approx \Theta\,\delta t + \frac{1}{2}\Gamma(\delta S)^2.$$

$$\mathbb{E}[\text{Hedged P\&L}] \approx \Theta\,\delta t + \frac{1}{2}\Gamma\,\mathbb{E}[(\delta S)^2] \approx \left(\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma\right)\delta t.$$

$$\frac{1}{2}\sigma^2 S^2 \Gamma = \frac{1}{2}\sigma^2 S^2 \frac{\phi(d_1)}{S\sigma\sqrt{T}} = \frac{S\phi(d_1)\sigma}{2\sqrt{T}} = -\frac{\partial C}{\partial t} = -\Theta.$$

$$\boxed{\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = 0}.$$

Put numbers on one trading day

At $S = K = \$100$, $T = 0.25$ year, $\sigma = 20\%$, with 252 trading days per model year:

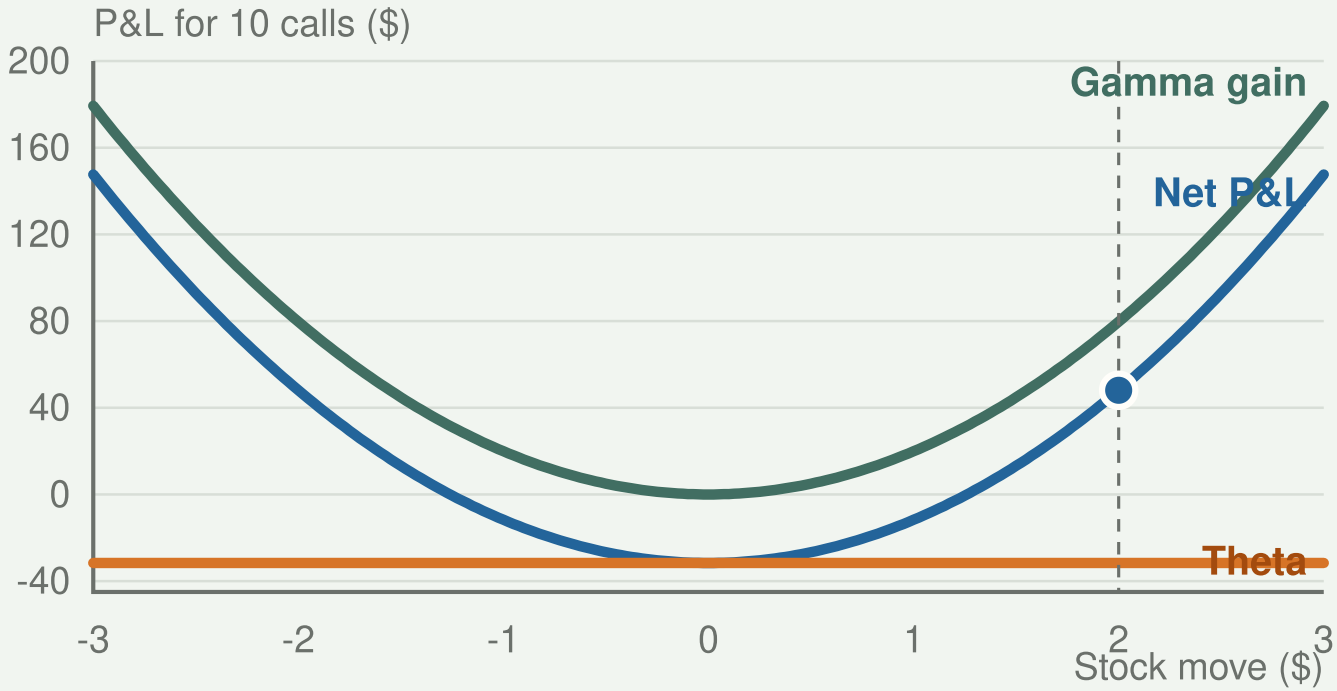
CALL PRICE	DELTA	GAMMA	THETA PER DAY
$\$3.988$	0.520	0.03984	$-\$0.0316$

$$\text{One-day hedged P\&L per share} \approx -\$0.0316 + \tfrac{1}{2}(0.03984)(\delta S)^2.$$

A stock move of about $\$1.26$ in either direction is the local break-even move. That is the model's one-day, one-standard-deviation move.

Try a stock move

Move the stock over one trading day. The option starts delta-hedged. The curves below are a **local approximation**, not exact repricing after a large move.



10 CALLS, 1,000 OPTION SHARES

Stock move	**+$2.0**
Gamma gain	**+$79.7**
Theta decay	**–$31.6**
Hedged P&L	**+$48.1**

Stock move ($)

–$3 $0 +$3

At zero move, the position bleeds. Large moves in either direction can overcome that bleed.

When does a delta hedge make money?

The option was priced using σ_{imp}. If the stock's actual short-interval volatility is σ_{realized}, its expected squared moves are:

$$\mathbb{E}[(\delta S)^2] \approx \sigma^2_{\text{realized}} S^2 \, \delta t.$$

$$\mathbb{E}[\text{hedged P\&L}] \approx \frac{1}{2} \Gamma S^2 \left(\sigma^2_{\text{realized}} - \sigma^2_{\text{imp}} \right) \delta t.$$

LONG OPTION

Likes realized movement above what its purchase price implied.

SHORT OPTION

Likes realized movement below what its sale price implied.

So far, we bought once and hedged stock

AT THE START

Buy or sell one option at an implied volatility.

ALONG THE PATH

Trade stock repeatedly to maintain the delta hedge.

AT THE END

The realized path decides how gamma gains compare with theta paid.

But an option itself has a market price **before** expiration. What if that quote changes while we still hold it?

Pause



Now trade the option's market price

We can also sell the option before expiry. Lecture 06 translated its market price into implied volatility, and that quote can change even if stock does not move.

STOCK HEDGE

Offsets the option's immediate exposure to S.

VOLATILITY QUOTE

Can reprice the option itself. A stock hedge does not remove this exposure.

We need a new sensitivity: how many dollars does the option price change for a change in quoted volatility?

Vega measures quote sensitivity

$$\nu = \frac{\partial C}{\partial \sigma}$$

ONE VOLATILITY POINT

From 20% to 21% means $\delta\sigma = 0.01$, not 1.

LONG OPTION

Positive vega means a higher implied-volatility quote raises its market value.

At fixed S, K, T, calls and puts have the same vega under put-call parity.

Derive vega from Black-Scholes

Hold S, K, and T fixed as the volatility input σ changes. We already know $S\phi(d_1) = K\phi(d_2)$ and $d_1 - d_2 = \sigma\sqrt{T}$.

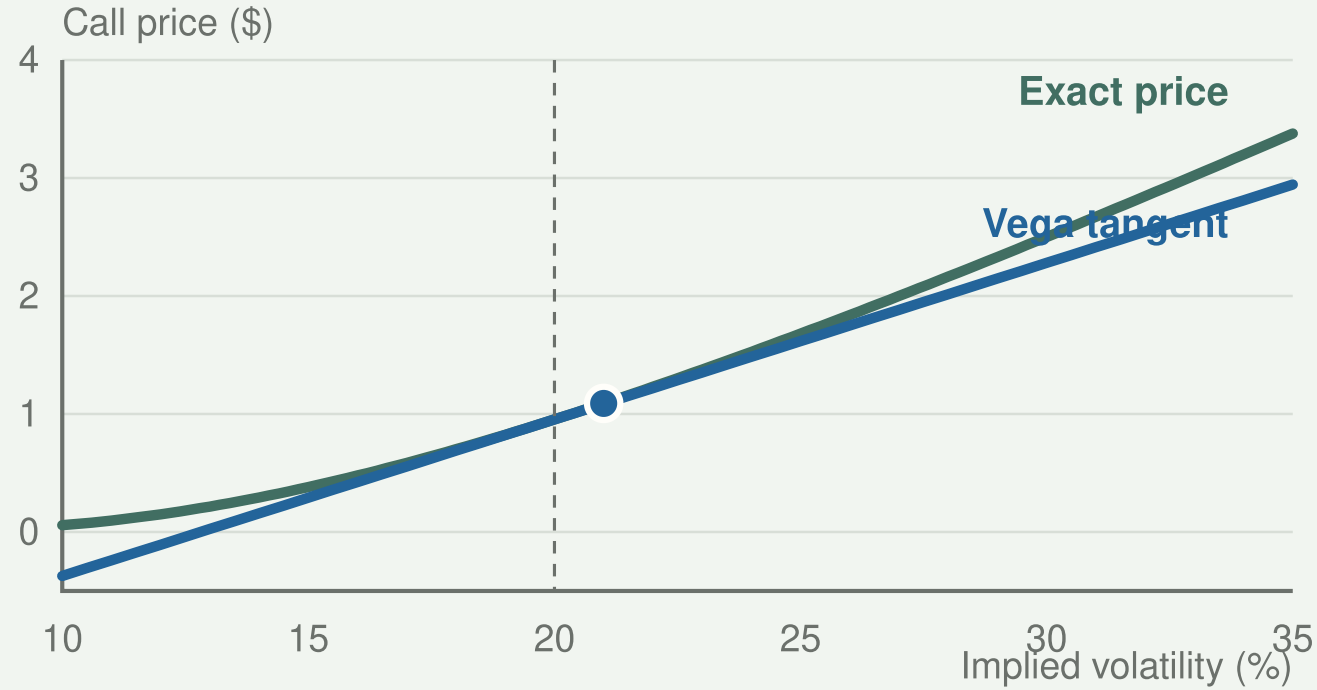
$$\nu = \frac{\partial C}{\partial \sigma} = S\phi(d_1)\frac{\partial d_1}{\partial \sigma} - K\phi(d_2)\frac{\partial d_2}{\partial \sigma}.$$

$$\nu = S\phi(d_1)\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right) = S\phi(d_1)\frac{\partial (d_1 - d_2)}{\partial \sigma}.$$

$$\boxed{\nu = S\phi(d_1)\frac{\partial (\sigma\sqrt{T})}{\partial \sigma} = S\phi(d_1)\sqrt{T} > 0}.$$

Move the volatility quote

Hold stock at $100, strike at $110, and three months to expiration. Start with a 20% quote, then move the quote.



10 CALLS, S = $100, K = $110

New IV	21%
New call price	$1.090
Exact P&L	+$135.6
Vega estimate	+$132.7

Implied volatility

10% 20% 35%

At 20% volatility, vega is about 13.27 per share per unit of volatility. A one-point rise adds roughly $0.133 per share, or $133 for ten contracts.

A crash put has two sources of P&L

Illustrative, exact Black-Scholes repricing. Buy 10 at-the-money puts at $S = \$100$, $T = 0.25$, 20% IV. Hedge their -0.480 delta by buying 480 shares.

STOCK FALLS TO $\$95$, IV STILL 20%

Put rises $\$3.988 \to \$6.888$. Stock hedge loses $\$2.400$ per put share. Net gain: $\$0.500$.

MARKET IV THEN RISES TO 25%

At $S = \$95$, put rises $\$6.888 \to \$7.762$. Additional gain: $\$0.874$ per share.

TOTAL, BEFORE COSTS AND ELAPSED TIME

Hedged gain $\approx \$1.374$ per share, or $\$1,374$ for 10 contracts.

The stock move creates convexity P&L. The quote change adds vega P&L. For a large move, exact repricing is safer than a local Greek approximation.

Combine delta, gamma, theta, and vega

For a short interval, zero rates and dividends, and a position hedged at its starting delta:

$$\text{Hedged P\&L} \approx \underbrace{\Theta\,\delta t}_{\text{time}} + \underbrace{\tfrac{1}{2}\Gamma(\delta S)^2}_{\text{stock movement}} + \underbrace{\nu\,\delta\sigma}_{\text{IV repricing}}.$$

Theta
What time costs

Gamma
What movement earns

Vega
What a new quote does

This is a local map, not an exact formula for jumps. Greeks change as stock, time, and IV change. Rates, dividends, and trading costs add more terms.

Time to expiry changes the P&L mix

Compare near-the-money options with similar other inputs. Hold the stock move, elapsed time, and IV quote change fixed.

SHORT TIME TO EXPIRY

$$\Gamma,\ |\Theta| \ \propto\ \frac{1}{\sqrt{T}}$$

Gamma P&L from stock movement and theta P&L from time passing grow. They tend to dominate the hedge's short-interval P&L.

LONG TIME TO EXPIRY

$$\nu \ \propto\ \sqrt{T}$$

The same IV quote change moves the option price more. Vega P&L becomes more important.

These are sensitivity comparisons, not guarantees. A large stock move or IV change can still dominate at any expiry.

Summary

1. A long option's changing delta makes the hedge sell after rises and buy after falls.
2. Gamma measures that convexity. Theta measures the cost of time passing.
3. At a fair Black-Scholes price, gamma gains and theta losses balance in model expectation.
4. A delta-hedged option is a trade on realized movement versus the movement its price implied.
5. Vega measures repricing when the market changes its implied-volatility quote.

Next: why implied volatility differs across strikes and expirations.