Black-Scholes

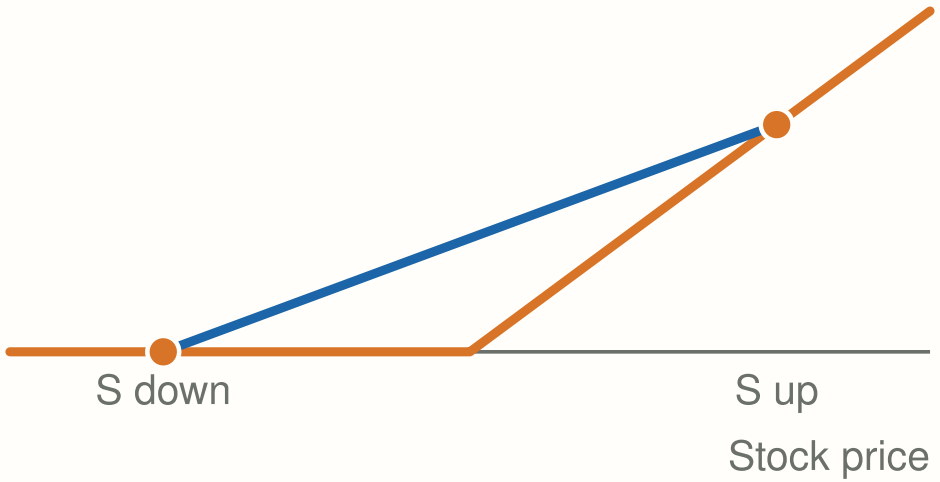
How much does a call cost?
Two routes to the same answer.

05

The same construction, with smaller moves

In Lecture 04, a stock-and-cash line matched **both possible outcomes**. Then we adjusted it after every move.

Lecture 04: two possible payoffs



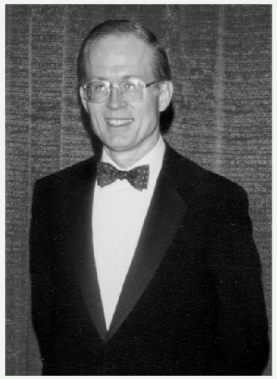
Next: two nearby values before expiry



Size of the next move ± 16 price units

As the moves get tiny, what happens to the line and the trading strategy?

Black, Scholes, and the paper



Fischer Black (1938–1995)

Economist and quantitative researcher. Co-developed the option-pricing model and worked at MIT and Goldman Sachs.



Myron Scholes (born 1941)

Economist. Shared the 1997 Economics Nobel with Robert Merton for work on derivatives pricing.

Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* **81**, no. 3 (May 1973), 637–654.

The contract and the trading rules

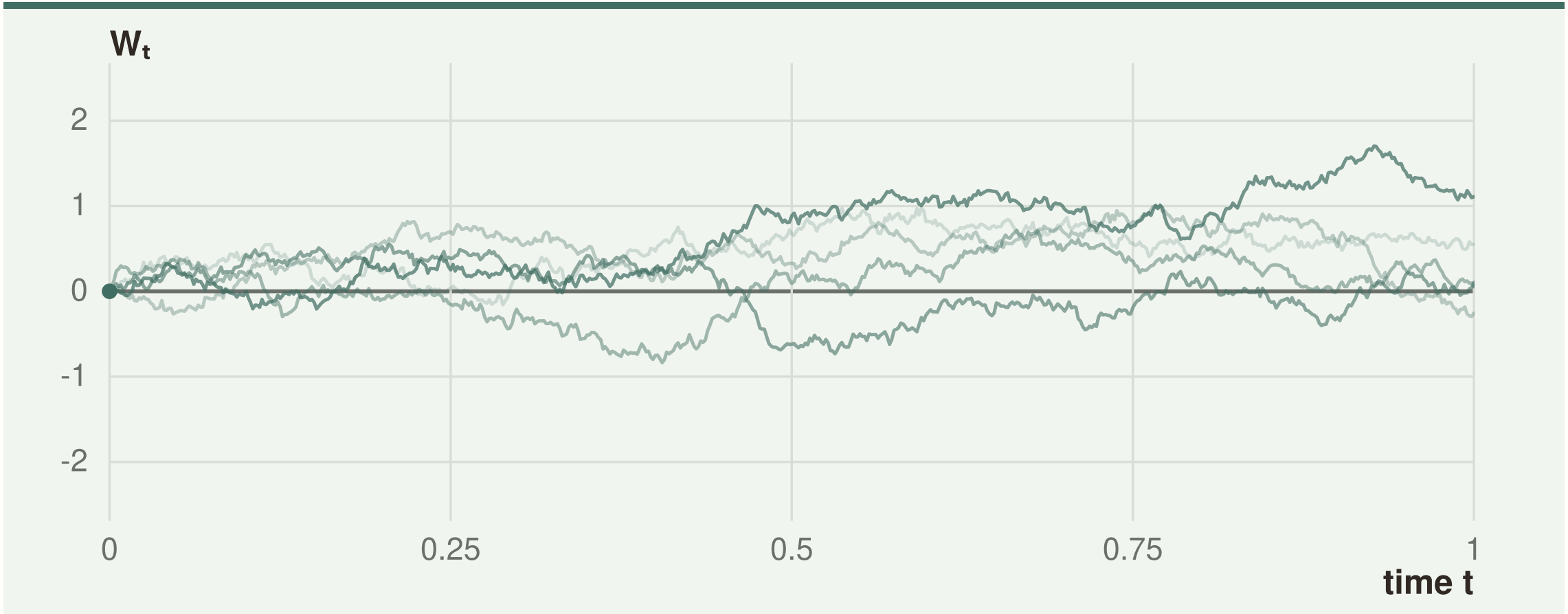
One European call. Strike K. Expiry T. No dividends.

1. We can trade fractional shares and borrow or lend cash.
2. Trading is continuous, with no transaction costs or stock-borrow constraints.
3. Borrowing and lending use the same constant rate r.

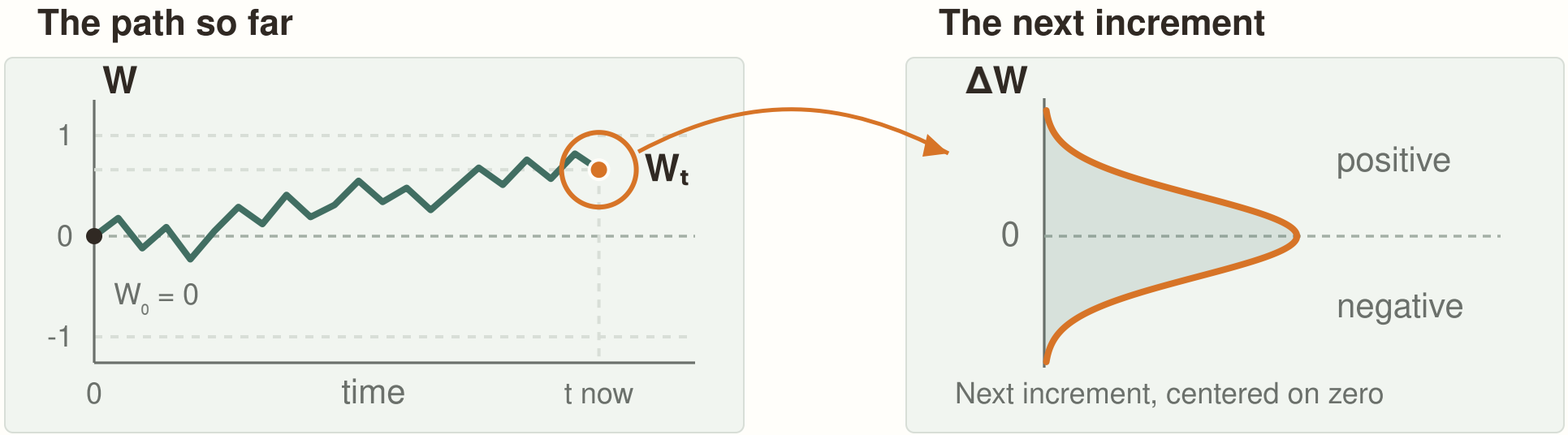
We still need a model for the stock's moves. That is our first step.

Step 1: The price process

Random Wiener walks



What can happen just after this point?



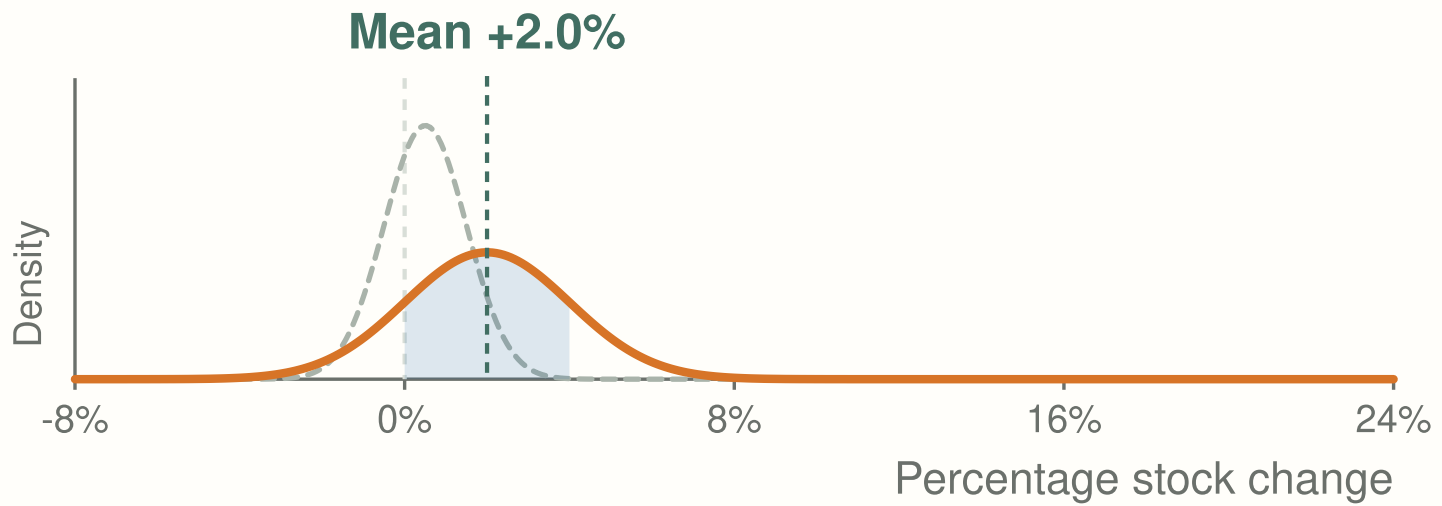
$$\Delta W = W_{t+\Delta t} - W_t \sim N(0, \Delta t)$$

Starts at $W_0 = 0$. Independent increments over non-overlapping intervals.

Variance $= \Delta t$. Standard deviation $= \sqrt{\Delta t}$.

A Wiener process (Brownian motion) is almost surely everywhere continuous and nowhere differentiable.

Drift and uncertainty grow at different speeds



Gray: initial interval. Shading: ±1 standard deviation.

Elapsed time 4 × initial interval

Elapsed time

× 4

Mean (drift)

+2.0% (×4)

Standard deviation

2.00% (×2.00)

$$\frac{\Delta S}{S} \approx \underbrace{\mu\,\Delta t}_{\text{average change}} + \underbrace{\sigma\sqrt{\Delta t}\,Z}_{\text{random change}}$$

$$Z \sim N(0,1)$$

$$\frac{\Delta S}{S} \sim N(\mu\Delta t, \sigma^2\Delta t)$$

$$\text{SD} = \sigma\sqrt{\Delta t}$$

Illustration: one initial interval has mean **+0.5%** and standard deviation **1%**.

In the formula, μ is expected return per year, σ is annualized volatility, and Δt is in years.

A model of percentage moves

Let the interval shrink: $\Delta S/S$ becomes dS/S. Multiply by S to express the move in dollars.

Percentage move

$$\frac{dS}{S} = \mu\,dt + \sigma\,dW$$

$\times S$ $\rightarrow$

Dollar move

$$dS = \underbrace{\mu S\,dt}_{\text{drift in dollars}} + \underbrace{\sigma S\,dW}_{\text{random move in dollars}}$$

1% of \$50 = **\$0.50**

1% of \$100 = **\$1.00**

This is **geometric Brownian motion**, with constant μ and known constant σ.
Its prices stay positive. Over a fixed interval, log returns are normal, so prices are lognormal.

dW is the continuous-time notation for the Wiener increment. These are real-world probabilities P; we will return to pricing probabilities later.

Why does a lognormal stock need a shift?

Equal up and down moves in *log price* do not give equal dollar moves. Start with $S_0 = 100$ and zero interest rates, so the expected expiry price must be 100.

Suppose $\sigma\sqrt{T} = 0.20$. Imagine two **equally likely log moves**, $+0.20$ and -0.20.

Up in log price

$$S_T = 100e^{0.20} \approx 122.14$$

+$22.14

Down in log price

$$S_T = 100e^{-0.20} \approx 81.87$$

–$18.13

The upside is larger than the downside. The average price is about **$102.01**, not $100.

For $\mathbb{E}^Q[S_T] = 100$, the *center of log returns* must shift down. The same effect holds for a continuous normal distribution.

How far down?

Write the normal log return as $m + aZ$, where $Z \sim N(0,1)$ and $a = \sigma\sqrt{T}$.

$S_T = S_0 e^{m+aZ}$

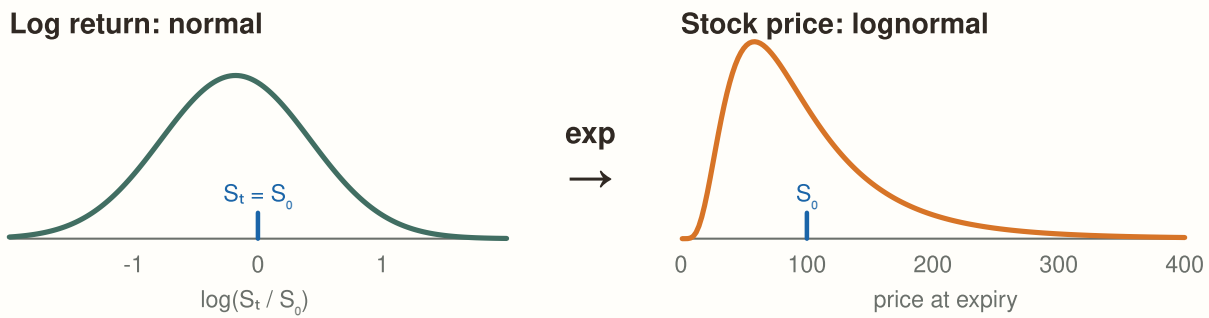
Because $\mathbb{E}[e^{aZ}] = e^{a^2/2}$,

$\mathbb{E}^Q[S_T] = S_0 e^{m+a^2/2} = S_0 e^{rT}$.

The exponents must match:

$m = rT - \frac{1}{2}a^2 = (r - \frac{1}{2}\sigma^2)T$.

$$S_T = S_0 \exp\left[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z\right]$$

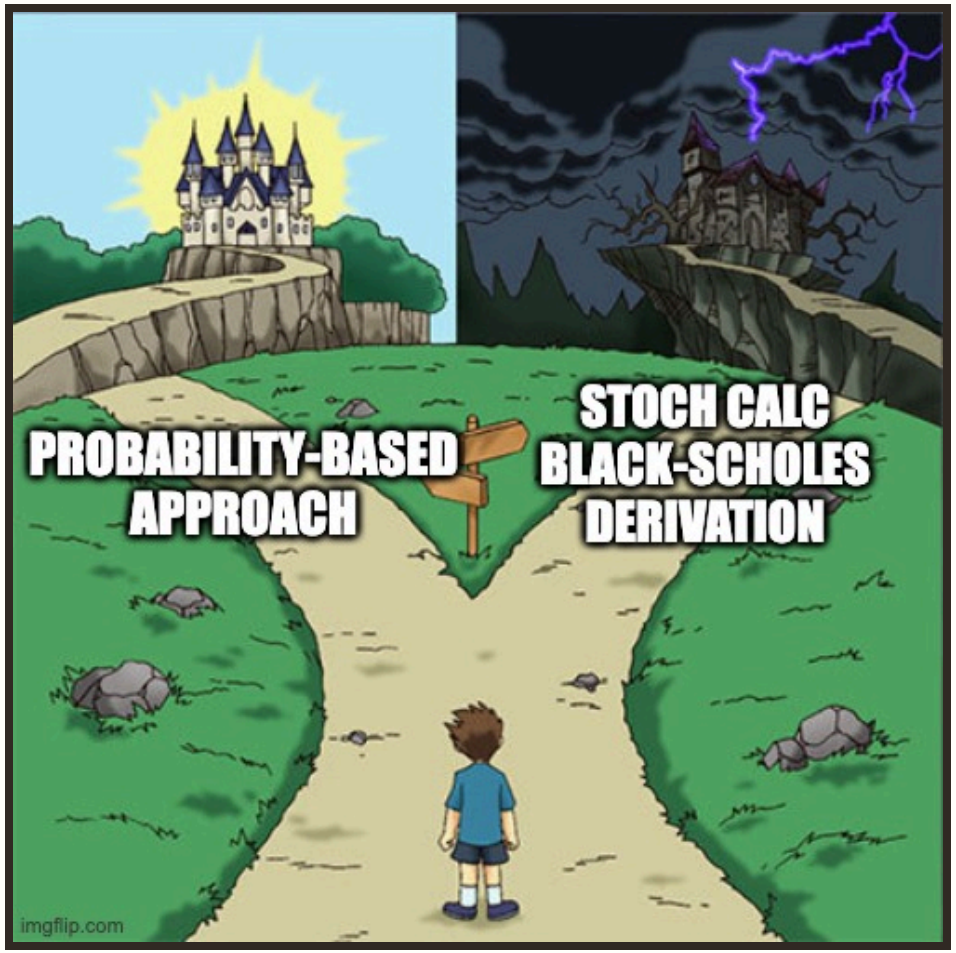


Under Q, not real-world P, expected stock growth is r. The $-\sigma^2/2$ shift corrects for lognormal asymmetry.

Pause



We have a choice



We will do both.

The stochastic calculus path…

Stock model: $dS = \mu S\,dt + \sigma S\,dW$. Let the unknown call price be $C(S,t)$.

Itô's step: The second-order term survives because $(dS)^2 \to \sigma^2 S^2\,dt$.

$$dC = C_t\,dt + C_S\,dS + \tfrac{1}{2}C_{SS}(dS)^2$$
$$= C_S\,dS + \left(C_t + \tfrac{1}{2}\sigma^2 S^2 C_{SS}\right)dt.$$

The partial derivatives

$$C_S = \frac{\partial C}{\partial S} \qquad C_{SS} = \frac{\partial^2 C}{\partial S^2} \qquad C_t = \frac{\partial C}{\partial t}$$

Replicate with stock and cash: $V = \delta S + B = C$, with $\delta = C_S$ and $B = C - \delta S$. Rebalance continuously, financing share trades through cash.

Portfolio's change: $dV = \delta\,dS + rB\,dt = C_S\,dS + r(C - SC_S)\,dt$.

No arbitrage: Match the call and portfolio changes. Their $C_S\,dS$ terms cancel, taking μ with them:

$$C_t + \tfrac{1}{2}\sigma^2 S^2 C_{SS} = r(C - SC_S).$$

$$\boxed{C_t + \tfrac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0}, \qquad C(S,T) = (S-K)^+.$$

Now all we have to do is solve a (homogeneous, linear) second-order PDE with certain boundary conditions. Or… we could take the more intuitive path.

Step 2: Price the payoff

Start from the expiry payoff

Lecture 04's no-arbitrage averaging rule says to discount the expected payoff.

Known payoff at expiry

$C_T = (S_T - K)^+$

Price today

$C_0 = e^{-rT}\mathbb{E}^Q[C_T]$

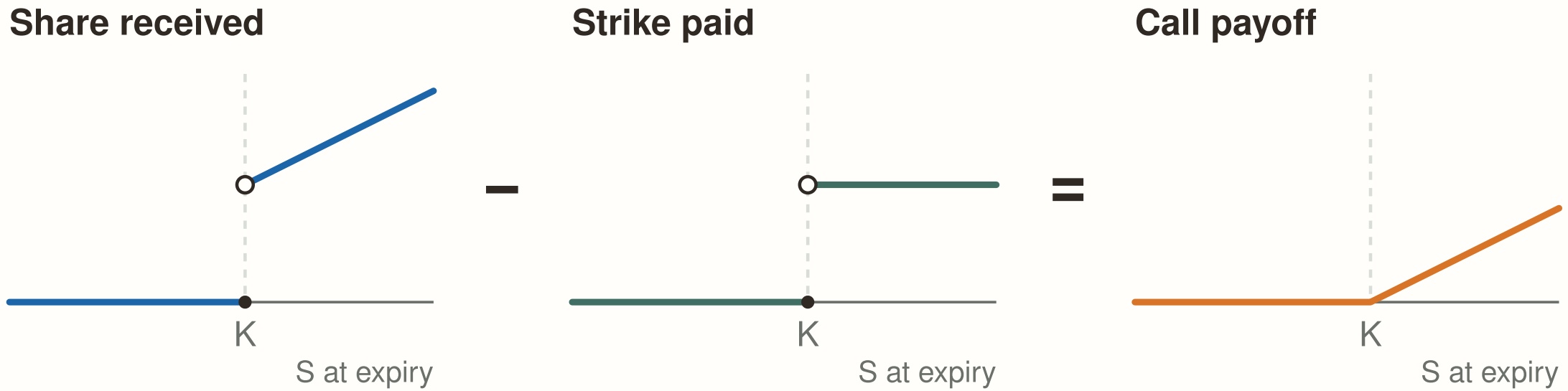
$$\boxed{C_0 = e^{-rT}\mathbb{E}^Q[(S_T - K)^+]}$$

And we know the distribution of S_T!

$$S_T = S_0 \exp\left[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z\right], \qquad Z \sim N(0,1).$$

Receive a share. Pay the strike.

Both cashflows happen only when $S_T > K$.

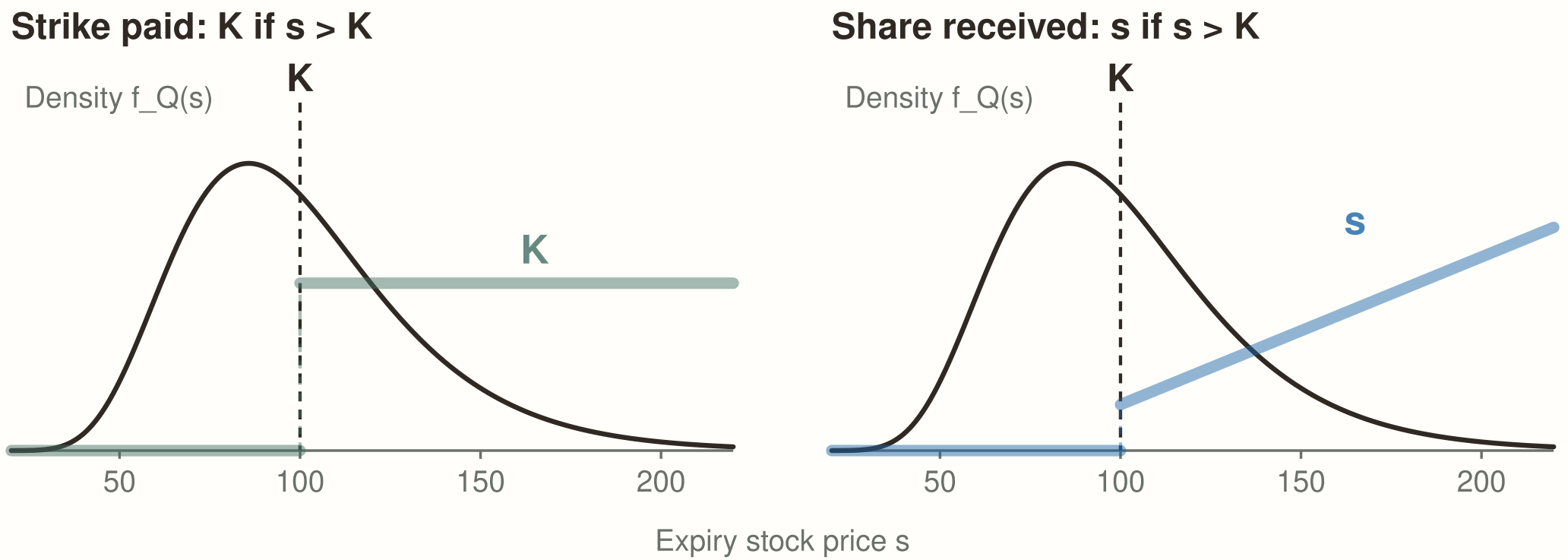


$$C_0 = \underbrace{e^{-rT}\mathbb{E}^Q[S_T \mathbf{1}_{\{S_T>K\}}]}_{\text{value of the share received}} - \underbrace{Ke^{-rT}Q(S_T > K)}_{\text{value of the strike paid}}$$

$\mathbf{1}_{\{S_T>K\}}$ is 1 in exercise states and 0 otherwise. We will calculate the cash piece first, then the share piece.

One distribution, two payments

The same possible expiry prices determine both pieces of the call. Each payment is zero below K.



The density $f_Q(s)$ gives the pricing weight of each expiry stock price s. The faint colored lines show what is paid at that price, using a separate vertical scale.

The fixed strike uses $N(d_2)$

The left graph pays a constant K whenever $S_T > K$. So we need the area under the density to the right of K.

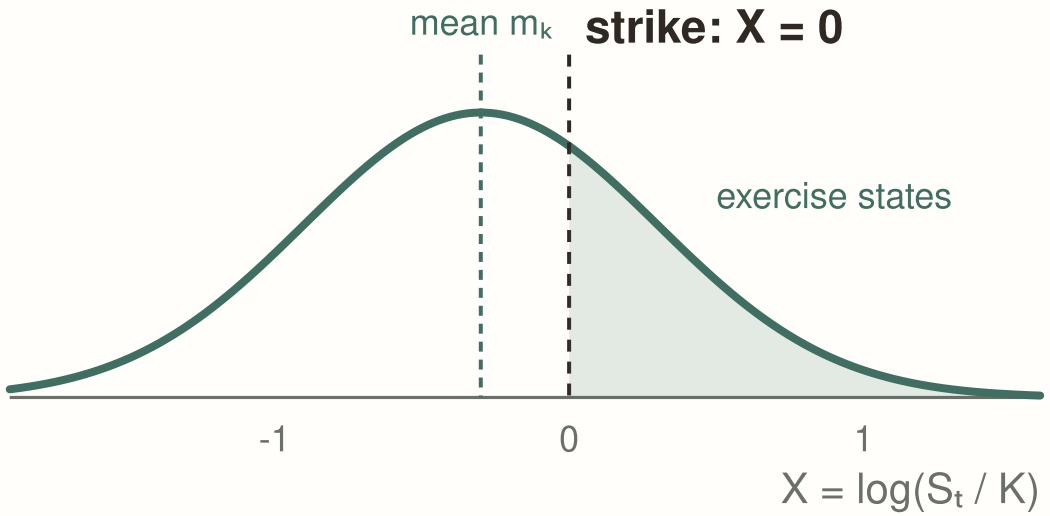


Illustration: the strike lies above the mean

From Slide 11, with $a = \sigma\sqrt{T}$:

$$X = \ln(S_T/K) = m_K + aZ$$

$$m_K = \ln(S_0/K) + (r - \tfrac{1}{2}\sigma^2)T$$

The strike sits at $Z = -m_K/a$. Define $d_2 = m_K/a$.

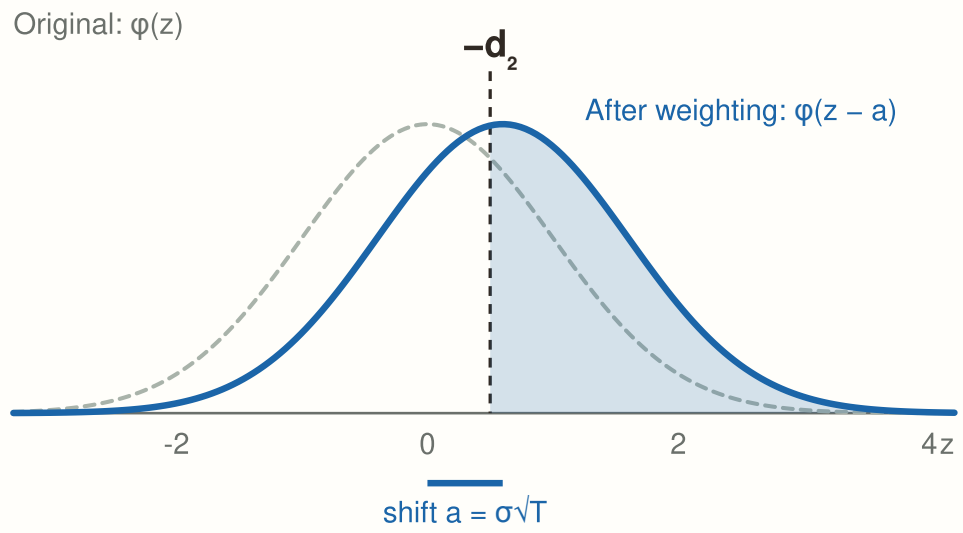
$$S_T > K \iff Z > -d_2$$

$$Q(S_T > K) = Q(Z > -d_2) = N(d_2)$$

$$V_{\text{strike}} = Ke^{-rT}\int_K^\infty f_Q(s)\,ds = Ke^{-rT}N(d_2)$$

The share received uses $N(d_1)$

The right graph pays the stock price s above K, not a constant. Start by multiplying the density by s.



$$V_{\text{share}} = e^{-rT} \int_K^\infty s f_Q(s)\, ds.$$

Set $a = \sigma\sqrt{T}$ and $s = S_0 e^{rT - a^2/2 + az}$. The cutoff stays $z = -d_2$.

Changing from s to z, and canceling $e^{-rT}e^{rT}$, gives

$$V_{\text{share}} = S_0 \int_{-d_2}^\infty e^{az - a^2/2} \phi(z)\, dz.$$

Complete the square in the exponent:

$$\begin{aligned} e^{az - a^2/2}\phi(z) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} + az - \frac{a^2}{2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-a)^2}{2}\right) = \phi(z-a). \end{aligned}$$

$$V_{\text{share}} = S_0 \int_{-d_2}^\infty \phi(z-a)\, dz = S_0 N(d_2 + a) = S_0 N(d_1), \qquad d_1 = d_2 + \sigma\sqrt{T}.$$

Put the two pieces together

$$C_0 = \underbrace{S_0 N(d_1)}_{\text{share received}} - \underbrace{K e^{-rT} N(d_2)}_{\text{strike paid}}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

A price, a stock holding, and a cash balance

$$S_0 = K = \$100, \quad r = 0, \quad T = \tfrac{1}{4}\ \text{year}, \quad \sigma = 20\%$$

$$\sigma\sqrt{T} = 0.10, \quad d_1 = 0.05, \quad d_2 = -0.05 \quad \Longrightarrow \quad C_0 \approx \$3.99$$

Stock

$51.994

0.51994 share.
$\delta = C_S = N(d_1)$

Cash

–$48.006

$B = C_0 - \delta S_0$.
Borrow this amount.

Net investment

$3.988

The same price as the call.

Differentiating the formula gives $C_S = N(d_1)$: the two density terms cancel.
As stock and time change, update the shares and finance the trade through cash.

We have answered the opening question: the model specifies both the initial cost and the continuing trading strategy. By parity, the matching at-the-money put also costs about $3.99.

What makes this a benchmark?

1 Exact model replication needs continuous trading. Real hedges are discrete, costly, and exposed to jumps.

2 We assumed a known constant σ. Future volatility is uncertain and changes over time.

3 Dividends, borrow, funding spreads, and early exercise can change the problem.

Next lecture: the market gives us a price. What volatility would make the formula reproduce it?

Summary

1. Percentage stock moves lead to normal log returns and lognormal prices.
2. The log-return center shifts down by $\sigma^2 T/2$ so the pricing distribution gives the right expected stock price.
3. Dynamic hedging produces the Black-Scholes pricing equation.
4. Discounted payoff averaging gives a second route to the same call price.
5. The formula gives both the price and the stock-and-cash hedge.

Next: turn observed option prices into implied volatility.