Pricing by replication

How do we manufacture an option payoff?

04

From parity to pricing

Lecture 03 showed that matching payoffs must have matching prices.

LAST TIME

A call minus a put is a financed share.

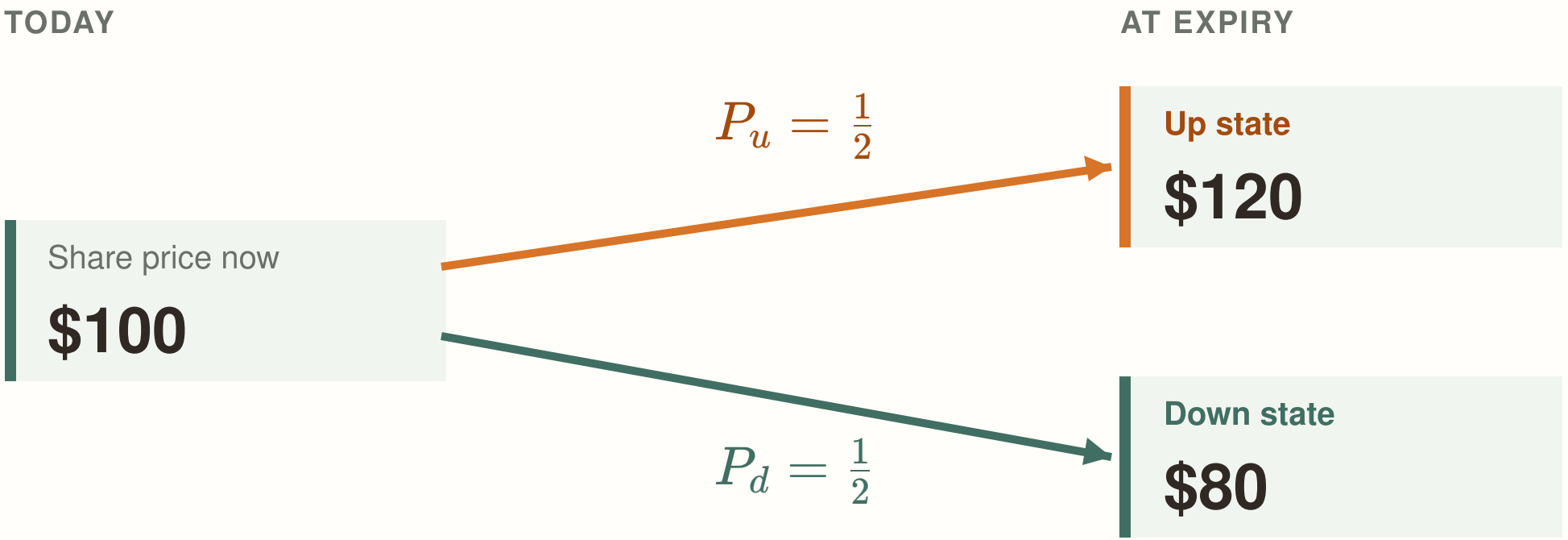
TODAY

Can stock and cash reproduce a call by themselves?

If we can build it, we can price it.

One share. Two possible tomorrows.

The share starts at **$100**. In this model, expiry has only two possible outcomes.



For now, interest is zero.

A kink, but only two possible endings

A strike-\$100 call has the usual kink. But this model allows only $S_d = \$80$ and $S_u = \$120$.



The blue straight line differs elsewhere, but goes through both outcomes that can actually occur.

Find the line yourself

Move Δ (shares) and B (cash) until the blue line hits both orange payoff dots.



Match both states

Portfolio payoff = $\Delta S_T + B$

Shares Δ	**0.50**
Cash B	**–$40**
At $80	**$0** Call: $0
At $120	**$20** Call: $20

Both outcomes match. You built the call.

The slope is the stock hedge

At zero interest, Δ shares plus B dollars of cash pay $\Delta S_T + B$ at expiry.

AT $S_d = \$80$

$$80\Delta + B = 0$$

AT $S_u = \$120$

$$120\Delta + B = 20$$

$$\underbrace{(120 - 80)}_{\text{share move}}\Delta = \underbrace{20 - 0}_{\text{call move}} \quad \Longrightarrow \quad \boxed{\Delta = \tfrac{1}{2}}$$

Half a share gains \$20 when a whole share gains \$40.

The cash leg sets the height and the price

With the slope fixed, use either outcome to place the line at the right height.

$$80\left(\tfrac{1}{2}\right) + B = 0 \quad \Longrightarrow \quad \boxed{B = -40}$$

Check the other point: $120\left(\tfrac{1}{2}\right) - 40 = \$20$. So the line matches in both possible states.

$$C_0 = \Delta S_0 + B = \tfrac{1}{2}(100) - 40 = \boxed{\$10}$$

Half a share and $40 of borrowing make the call. Identical payoffs must have the same price.

Why does the replicating portfolio cost $10?

$B = -40$ means borrowing $40. The portfolio is a pair of trades we can make today.

BUY HALF A SHARE

At $100 per share, half a share costs $\frac{1}{2}(100) = 50$ dollars today.

BORROW CASH

Receive **$40 today**. At zero interest, repay **$40 at expiry**.

$$\text{Net cost today} = \$50 - \$40 = \boxed{\$10}$$

At expiry, after repaying the loan: down $\frac{1}{2}(80) - 40 = 0$; up $\frac{1}{2}(120) - 40 = 20$.

Those are exactly the call's payoffs. By no arbitrage, the call must also cost **$10**.

Spot the arbitrage

The call trades at **$9**. Its replicating portfolio costs **$10**. What trade pays you today and has no expiry risk?

BUY CHEAP

Buy the call for **$9**.

SELL EXPENSIVE

Short the replicating portfolio for **$10**.

Collect **$1 now**. At expiry the two payoffs cancel in both states.

What “risk-neutral” means

It is a **no-arbitrage pricing rule**, not a claim that people are indifferent to risk.

Q · PRICING WEIGHT

Choose q so the stock’s price equals the weighted average of its outcomes at zero interest:

$$100 = q(120) + (1 - q)(80) \quad \Rightarrow \quad q = \tfrac{1}{2}$$

That same weight prices any payoff replicated by stock and cash.

P · REAL-WORLD VIEW

How likely is each outcome *actually*? Investors need not act as if every dollar is equally valuable.

With **concave utility**, a loss can hurt more than an equal gain helps. Investors, and markets in aggregate, may pay for crash protection.

Q makes traded prices consistent with replication. P describes beliefs about what will happen. They need not agree.

Crash insurance: P and Q can differ

A broad index is **100 today**. Next period it is **110** in a calm market or **70** after a crash. A strike-90 put pays **20** only in the crash. Assume zero interest.

P · FORECAST

Suppose investors estimate a **10%** crash chance.

Their expected index level is $0.90(110) + 0.10(70) = 106$, above its **100** price.

$$\mathbb{E}^P[\text{put payoff}] = 0.10(20) = \$2$$

That is the expected payoff, **not** the put's price.

Q · NO-ARBITRAGE PRICE

The traded index price implies a larger crash weight:

$$Q(\text{crash}) = \frac{110 - 100}{110 - 70} = 25\%$$

$$\text{put price} = 0.25(20) = \$5$$

The **$5** is a replication price, not a 25% crash forecast. Risk aversion can help explain why crash protection costs more than its forecast payoff.

Bring interest back

$$q = \frac{S_0 e^{rT} - S_d}{S_u - S_d}, \qquad C_0 = e^{-rT}\left[qC_u + (1-q)C_d\right]$$

Under the risk-neutral distribution, the expected stock price at expiry is $qS_u + (1-q)S_d = S_0 e^{rT}$. The cash leg grows at the same risk-free rate.

We assume no dividends, frictionless stock trading, and one borrowing/lending rate.

When does the model avoid arbitrage?

$$S_d < S_0 e^{rT} < S_u \quad \Longleftrightarrow \quad 0 < q < 1$$

If the risk-free growth of the share's starting price lies outside the two outcomes, stock and cash already offer an arbitrage.

The familiar "probability" range is a consequence of a no-arbitrage condition.

Price the matching put

Return to $S_d = \$80$, $S_u = \$120$, $K = \$100$, and zero rates.

DOWN

Put payoff **$20**

UP

Put payoff **$0**

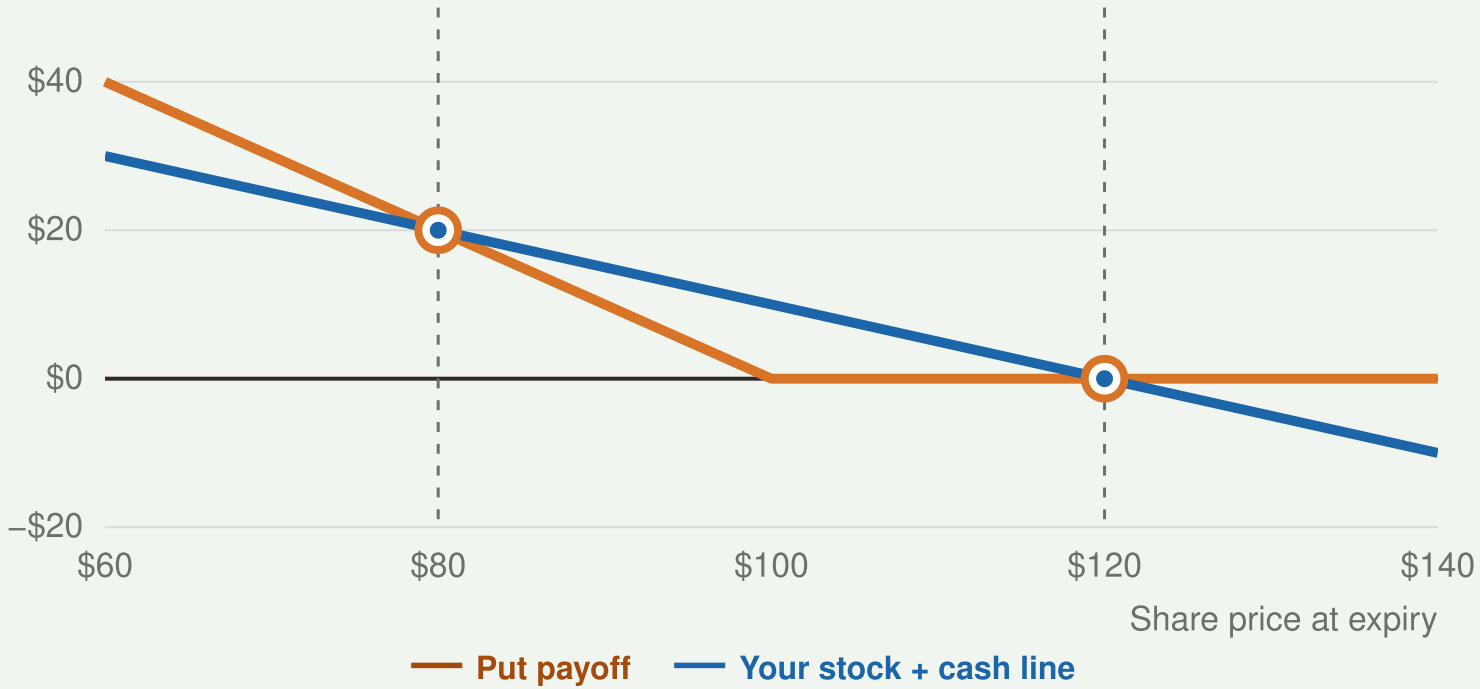
$$\Delta_P = \frac{0-20}{120-80} = -\frac{1}{2}$$

$$20 = 80\left(-\tfrac{1}{2}\right) + B \quad \Longrightarrow \quad B = 60$$

$$P_0 = \Delta_P S_0 + B = -\tfrac{1}{2}(100) + 60 = \boxed{\$10}$$

Build the put yourself

Now match the put's **$20 down** and **$0 up** with shares Δ and cash B.



Match both states

Portfolio payoff = $\Delta S_T + B$

Shares Δ	**-0.50**
Cash B	**$60**
At $80	**$20**
	Put: $20
At $120	**$0**
	Put: $0

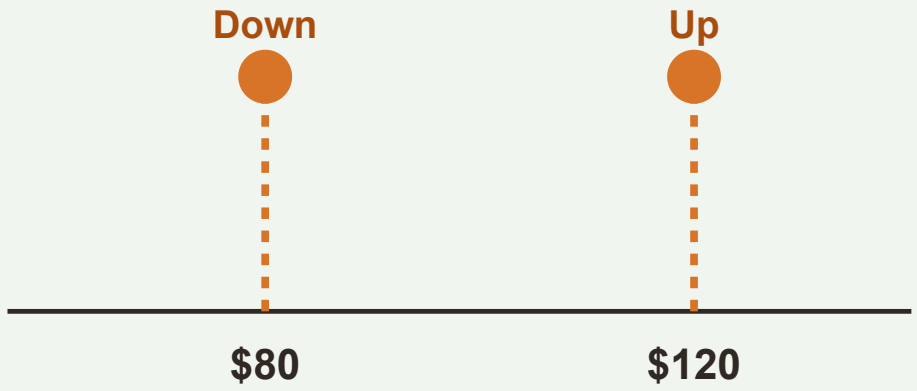
Both outcomes match. You built the put.

Time for a pause.

But does this actually work?

OUR ONE-STEP MODEL

Only two possible prices



A RICHER EXPIRY DISTRIBUTION

Prices across a continuous range

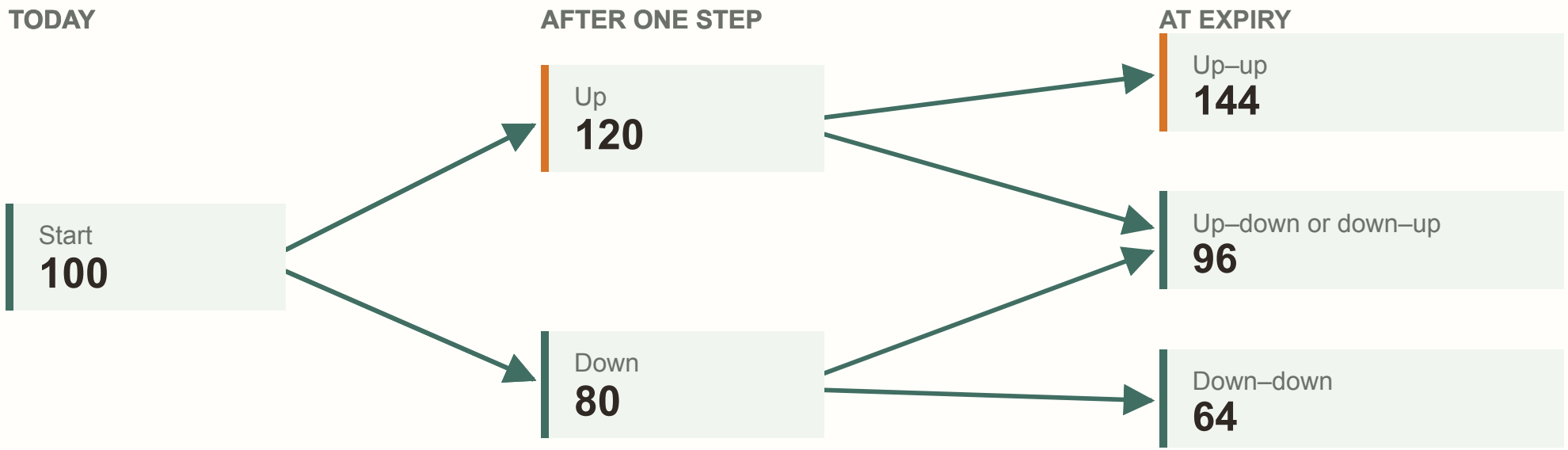


How can a two-outcome model say anything useful about that richer world?

> **The trick:** chop time into much smaller steps. At each step, there are just two nearby moves. Then we can adjust the hedge again.

From one to two steps

Now allow a move **today to tomorrow**, then another move **tomorrow to expiry**.



Each step multiplies the share by 1.2 or 0.8. The middle paths meet at 96.

Start at expiry and work backward

For a $K = \$100$ call, terminal payoffs at **144** / **96** / **64** are **44** / **0** / **0**.

AT THE 120 NODE

Next values are 44 and 0. Call value: **22**.

AT THE 80 NODE

Next values are 0 and 0. Call value: **0**.

$$C_0 = \tfrac{1}{2}(22) + \tfrac{1}{2}(0) = \boxed{\$11}$$

The replication changes…

Today the hedge holds **0.55 shares** (cost $55) and owes **$44**. Net cost: **$11**. After tomorrow's move, we must trade again.

IF THE STOCK RISES TO 120

The next day's hedge needs $\Delta_u = \frac{44-0}{144-96} = \frac{11}{12} \approx 0.917$ shares.

Buy 0.367 more shares for $44. Borrow another $44 to pay for them.

IF THE STOCK FALLS TO 80

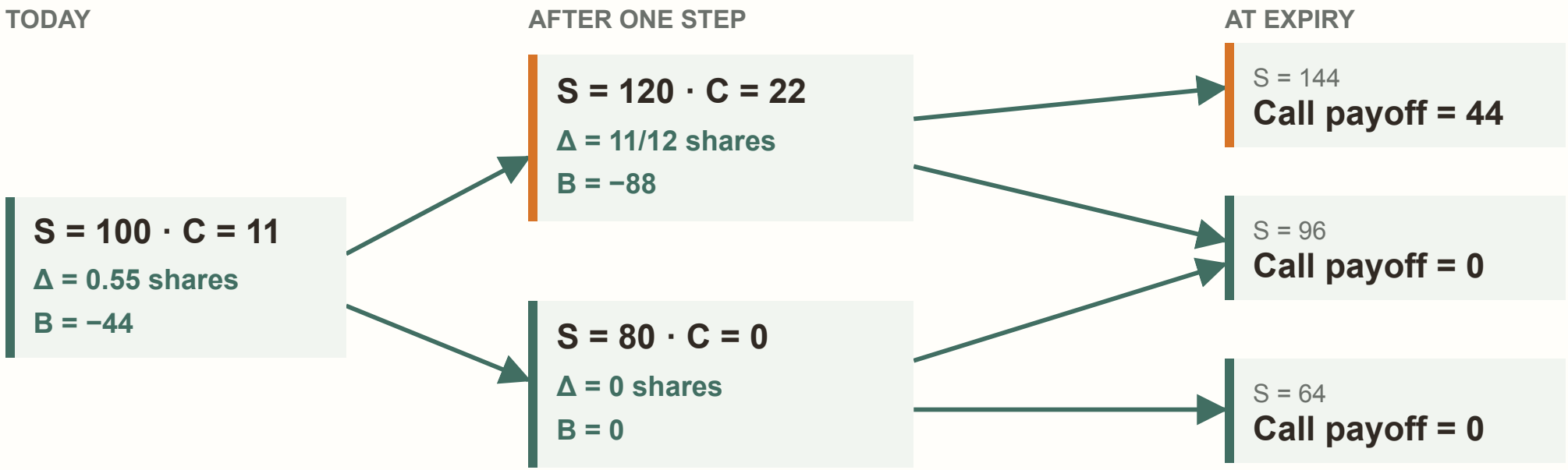
The next day's hedge needs $\Delta_d = 0$ shares.

Sell all 0.55 shares for $44. Use the proceeds to repay the $44 loan.

To maintain replication, **rebalance both stock and cash after every daily move**. This is a trading strategy, not a buy-and-hold portfolio.

The hedge at every node

For the strike-$100 call, each pre-expiry node has its own shares Δ and cash B. Negative B means borrowing.



After the first move, replace today's hedge with the one at the new node.

More steps, smaller moves

In a standard binomial tree, a short step of length Δt uses

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

SHORTER TIME STEP

Smaller typical price moves.

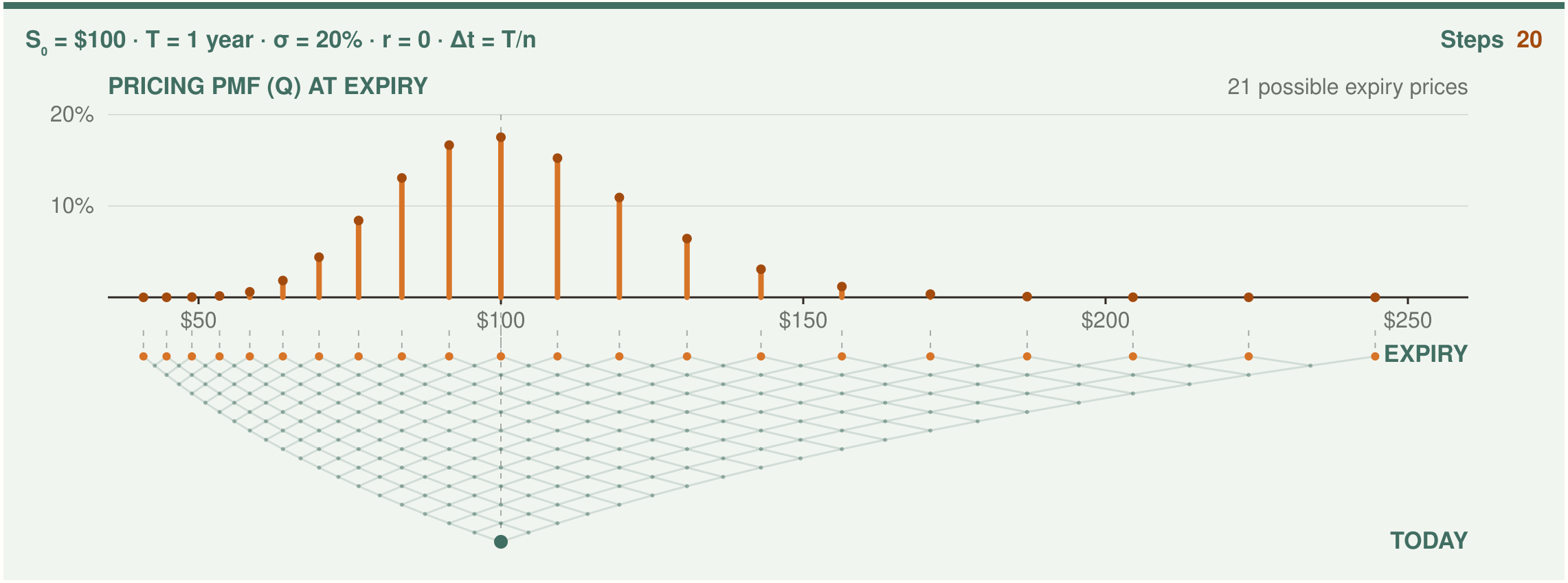
MORE HEDGE DATES

The portfolio can be adjusted more often.

This is a *new* tree parameterization, not the fixed 1.2 / 0.8 example above. At zero rates, u and d no longer average to 1, so the pricing probability is $q \neq \frac{1}{2}$.

From a tree to a distribution

Follow the recombining tree upward. Each expiry-price bar sums the probabilities of all paths that reach it.



The bars use the no-arbitrage q at each step. At 20 steps there are still 21 outcomes; as the steps get finer, the distribution approaches a continuous log-normal shape.

Trees can also handle early exercise

At each node of an American option, compare exercising now with continuing.

$$V_{\text{node}} = \max\left(\text{exercise value},\ \text{discounted continuation value}\right)$$

The same backward tree can price European options or add the American exercise choice.

The binomial model is elegant and flexible

WHAT MAKES REPLICATION WORK:

- Trade stock and cash at every node.
- Borrow and lend at one interest rate.
- Specify the two possible stock moves at each step.

REAL MARKETS MEAN HANDLING LOTS OF DETAILS:

- Bid–ask spreads and transaction costs.
- Stock borrow costs and availability.
- Dividends and corporate actions.
- Changing volatility and price jumps.
- Discrete hedge times.
- Day boundaries and overnight gaps.

However it’s more computationally heavyweight than a closed-form equation. That’s what Black-Scholes promises.

Summary

1. In a two-state model, stock and cash can exactly reproduce an option payoff.
2. Matching payoffs force the option price to equal the replication cost.
3. Risk-neutral probabilities are a shortcut derived from that hedge, not forecasts.
4. In a multi-step tree, the stock hedge changes after each move.
5. Smaller steps and more frequent rehedging point toward a continuous-time model.

Next: derive Black-Scholes by doing this in the continuous limit.