Solutions to Tutorial 7

1 Why did my call lose value?

Imagine that you own a European call option.

(a) All else being equal, the call tends to be worth less as time passes. Explain why.

(b) On one particular day, the price at which you could sell the call falls much more than ordinary time decay would suggest. What might have happened?

The call gives its owner the right, but not the obligation, to buy the stock for K at expiry. With more time remaining, there are more opportunities for the stock price to move far above the strike. This possibility has value even when the option is currently out of the money. As expiry approaches, fewer favorable moves remain possible and this *time value* disappears. At expiry, the call is worth only its intrinsic value,

$$(S_T - K)^+.$$

Thus, if the stock price, volatility, interest rate, and all other inputs are held fixed, a standard long call generally loses value as calendar time passes. This is the effect measured by a negative theta.

A much larger one-day loss need not have been caused by time alone. Possible explanations include:

- the stock price fell, reducing the value of a call with positive delta;
- implied volatility fell, perhaps because a major announcement took place and uncertainty disappeared (a *volatility crush*);
- the call moved close to expiry, where time decay can be especially rapid for an option near the money;
- liquidity deteriorated or the bid–ask spread widened, lowering the bid at which the option could actually be sold.

The last point matters because the question concerns an executable market price, not only a theoretical model value.

2 Which Greek sign combinations are possible?

At zero interest rates and with no dividends, the value of a European call with time to expiry T is

$$C(S,T) = SN(d_1) - KN(d_2), \qquad d_1 = \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

where N is the standard normal distribution function and $\phi = N'$ is its density. Its Greeks are

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) > 0, \qquad \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{T}} > 0, \qquad \Theta = -C_T = -\frac{S\phi(d_1)\sigma}{2\sqrt{T}} < 0.$$

For each possible combination of positive or negative delta, gamma, and theta, construct a portfolio of European options if possible and state the conditions under which the construction works. Options at different strikes may have different implied volatilities.

The following plots first show how the Black–Scholes call-price curve changes when one parameter is varied and the others are held fixed. The interest rate is zero in every panel.

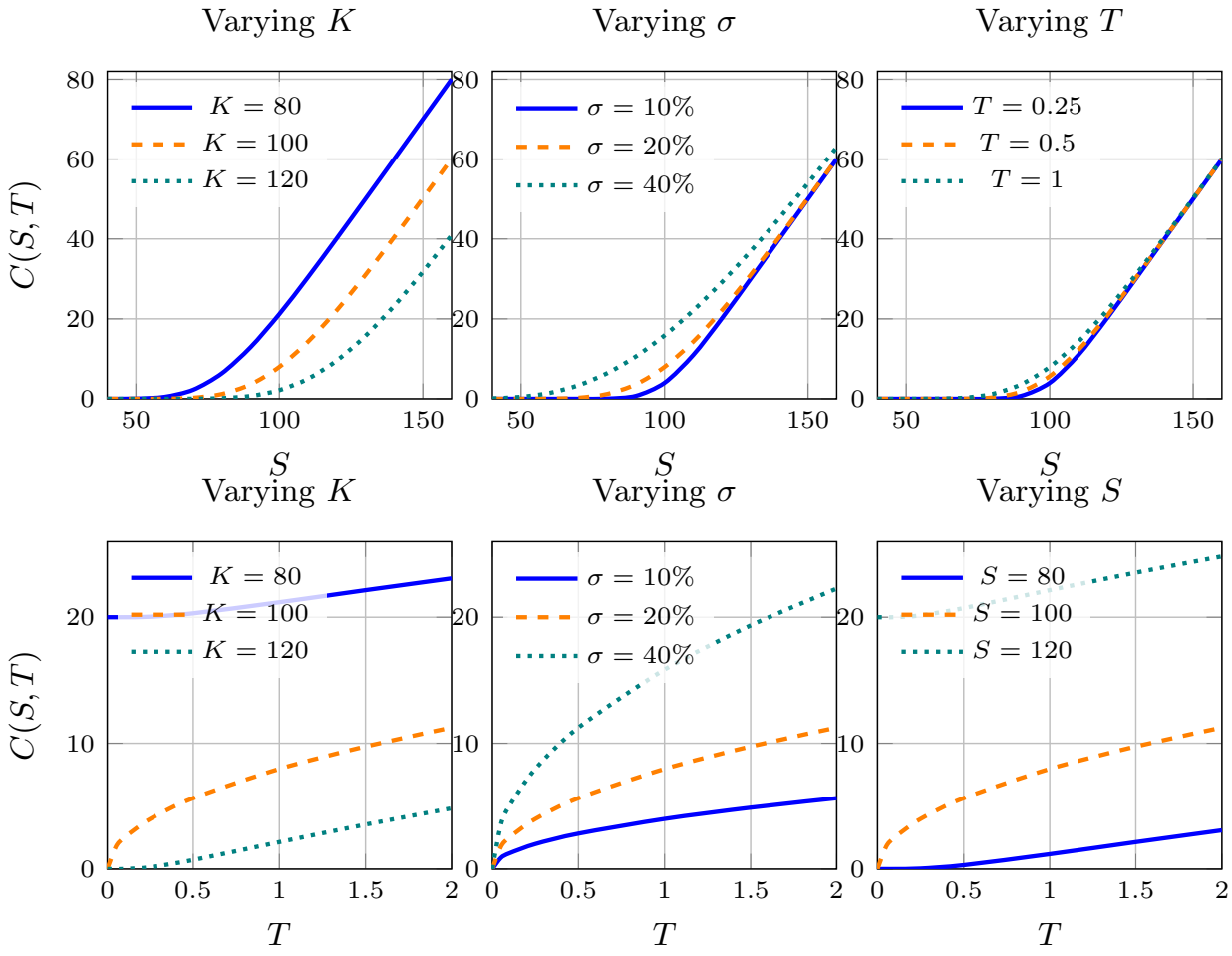

Figure 1: Black–Scholes call values. The upper row plots the value against S: from left to right, the panels vary K, σ, and T. The lower row plots the value against T: the panels vary K, σ, and S. Parameters not shown in a legend are fixed at $S = K = 100$, $\sigma = 20\%$, and $T = 1$, as applicable.

Every curve is increasing and convex in S, illustrating positive delta and positive gamma. A higher strike lowers the call value. Increasing volatility raises the value by increasing the benefit of convexity. More time to expiry also raises the value here. Since calendar time passing reduces T, this last observation corresponds

to negative theta. The lower row makes the time effect explicit: at $T = 0$ the curves begin at intrinsic value, and additional time increases the call value.

Greeks are additive: the Greek of a portfolio is the sum of the Greeks of its positions. First observe that if every option is priced with the same volatility σ, then

$$\frac{-\Theta}{\Gamma} = \frac{1}{2}S^2\sigma^2$$

is the same for every strike. Any portfolio then satisfies

$$\boxed{\Theta = -\frac{1}{2}\sigma^2 S^2 \Gamma.}$$

Under a flat volatility, gamma and theta must have opposite signs. The four immediate constructions follow from put–call parity, $P = C - S + K$: a put has delta $N(d_1) - 1 < 0$ and the same gamma and theta as a call. Therefore

Δ	Γ	Θ	Construction
$+$	$+$	$-$	Long call
$+$	$-$	$+$	Short put
$-$	$+$	$-$	Long put
$-$	$-$	$+$	Short call

The conclusion changes when implied volatility depends on strike. Consider two calls C_1 and C_2, and the portfolio

$$V = C_1 - nC_2, \qquad n > 0.$$

Write

$$d_i = \Delta_i > 0, \qquad g_i = \Gamma_i > 0, \qquad q_i = -\Theta_i > 0,$$

and define the three thresholds

$$D = \frac{d_1}{d_2}, \qquad G = \frac{g_1}{g_2}, \qquad Q = \frac{q_1}{q_2}.$$

The portfolio Greeks are

$$\Delta_V = d_1 - nd_2, \qquad \Gamma_V = g_1 - ng_2, \qquad \Theta_V = -q_1 + nq_2.$$

Their signs change when n crosses D, G, and Q, respectively. Hence the exact conditions for the eight combinations are

Δ_V	Γ_V	Θ_V	Condition on n
$+$	$+$	$+$	$Q < n < \min(D, G)$
$+$	$+$	$-$	$0 < n < \min(D, G, Q)$
$+$	$-$	$+$	$\max(G, Q) < n < D$
$+$	$-$	$-$	$G < n < \min(D, Q)$
$-$	$+$	$+$	$\max(D, Q) < n < G$
$-$	$+$	$-$	$D < n < \min(G, Q)$
$-$	$-$	$+$	$n > \max(D, G, Q)$
$-$	$-$	$-$	$\max(D, G) < n < Q$

Each row is possible for this pair of calls precisely when the displayed interval is nonempty.

In particular, a $+++$ portfolio requires

$$\boxed{Q < n < \min(D, G).}$$

Suppose C_2 is the higher-strike call. Its delta can be very small, making $D = \Delta_1/\Delta_2$ large. Moreover,

$$q_i = \frac{1}{2}S^2\sigma_i^2 g_i, \qquad Q = \frac{\sigma_1^2}{\sigma_2^2}G.$$

Thus $Q < G$ when the higher-strike call has the larger implied volatility, $\sigma_2 > \sigma_1$. If also $Q < D$, one may short a number n of the higher-strike calls that is large enough to make theta positive but not so large that delta or gamma becomes negative. This produces the proposed $+++$ position.

3 Hedged P&L over two days

For a delta-hedged option position, use the approximation

$$\text{Hedged P\&L} \approx \Theta\,\delta t + \frac{1}{2}\Gamma(\delta S)^2 + \nu\,\delta\sigma,$$

where ν is the sensitivity to implied volatility. A call has theta -0.01 dollars per day and gamma 0.03 dollars per squared-dollar move. Over two consecutive days, the stock first rises by one dollar and then falls by one dollar. Assume that the Greeks remain constant, implied volatility does not change, and the hedge is rebalanced daily. Compute the hedged P&L on each day and in total.

Implied volatility is unchanged, so $\delta\sigma = 0$. On the first day, $\delta t = 1$ and $\delta S = +1$, hence

$$\text{P\&L}_1 \approx -0.01 + \frac{1}{2}(0.03)(1)^2 = 0.005 \text{ dollars}.$$

On the second day, $\delta S = -1$. The stock contribution is unchanged because the move is squared:

$$\text{P\&L}_2 \approx -0.01 + \frac{1}{2}(0.03)(-1)^2 = 0.005 \text{ dollars}.$$

The total hedged P&L is therefore

$$\boxed{\text{P\&L}_{\text{total}} \approx 2(-0.01) + \frac{1}{2}(0.03)\big(1^2 + (-1)^2\big) = 0.01 \text{ dollars}.}$$

Although the stock finishes where it started, the long-gamma position benefits from movement in either direction. In this example, the gamma gains slightly exceed the two days of time decay. Transaction costs and changes in the Greeks are ignored.