Tutorial 5

1 Newton's method by hand

Starting from $x_0 = 1$, use Newton's method to find the positive solution of

$$\cos x = x^3$$

by computing x_1, x_2, and x_3. Use the table at the end of the sheet for the required numerical values.

2 Newton's method can cycle

Apply Newton's method to

$$f(x) = x^3 - 2x + 2$$

with $x_0 = 0$. Compute x_1, x_2, x_3, and x_4.

3 Computing implied volatility

A non-dividend-paying stock has current price $S_0 = 100$. Consider a European call with strike $K = 100$, one year to expiry, and a continuously compounded risk-free rate $r = 5\%$. Its observed market price is $C_{\text{mkt}} = 10.4506$.

For volatility $\sigma > 0$, the Black–Scholes call price is

$$C_{\text{BS}}(\sigma) = S_0\Phi(d_1(\sigma)) - Ke^{-rT}\Phi(d_2(\sigma)),$$

where

$$d_1(\sigma) = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2(\sigma) = d_1(\sigma) - \sigma\sqrt{T},$$

and Φ is the standard normal distribution function.

Formulate the implied-volatility calculation as a root-finding problem for Newton's method. Specify the function, its derivative, and the iteration that we could give to a numerical solver. Use the initial guess $\sigma_0 = 0.30$ and report the resulting implied volatility.

4 Using second derivatives

Let $f \in C^2$. Derive Halley's method for finding a root of f:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}.$$

Reference values for the hand calculation

Values are rounded to six decimal places.

n	x_n	$\cos x_n$	$\sin x_n$	x_n^2	x_n^3
0	1.000000	0.540302	0.841471	1.000000	1.000000
1	0.880333	0.636895	0.770951	0.774986	0.682246
2	0.865684	0.648119	0.761539	0.749409	0.648752
3	0.865474	0.648279	0.761403	0.749045	0.648279