Assignment 2: Volatility and hedging

Lectures 6–8. Show your reasoning, positions, and units. Options are European and written on one share. Rates, dividends, and trading costs are zero throughout. You may use a calculator or spreadsheet, but no programming is required. Unless a question says otherwise, treat the stated Greeks as local sensitivities, not exact predictions after a large move.

1 From option prices to a stock hedge [30 points]

A stock trades at $S = \$100$. Two options expire in $T = 0.25$ years:

Option	Strike	Market price
Call	\$95	\$6.89
Put	\$105	\$8.95

For this question, use $C = SN(d_1) - KN(d_2)$, $P = C - S + K$, and

$$d_1 = \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Here N is the standard normal cumulative distribution function. The following Black–Scholes prices, in dollars, are provided so that the numerical inversion can be done on paper:

Volatility	95-strike call	105-strike put
15%	6.07	6.19
20%	6.89	7.06
25%	7.76	7.99
30%	8.67	8.95
35%	9.59	9.92

(a) Find the implied volatility of each quoted option. Why is it legitimate for these two options, which have different strikes, to have different implied volatilities?

(b) You own six of the calls and four of the puts. Compute each option's delta using $\Delta_C = N(d_1)$ and $\Delta_P = N(d_1) - 1$. Find the portfolio delta and state the stock trade that makes it delta-neutral. Numerical values: $N(0.5629) = 0.7133$ and $N(-0.2503) = 0.4012$.

(c) If the stock price changes tomorrow but you do not trade again, must the portfolio still be delta-neutral? Explain using gamma, without calculating a new option price.

2 One option and one share trade [35 points]

At $S = \$100$, you hold the following three-option portfolio. All options have the same three-month expiry and are valued at 20% volatility. The table gives Greeks *per one-share option.* Vega ν is the value change for a *one-percentage-point* rise in implied volatility.

Option	Position	Δ	Γ	ν
90-strike call	+2	+0.86512	0.02170	0.10849
100-strike put	−1	−0.48006	0.03984	0.19922
110-strike call	+1	+0.18324	0.02653	0.13267

You may trade any quantity, including a fractional quantity, of one additional 100-strike call. It has $\Delta = 0.51994$, $\Gamma = 0.03984$, and $\nu = 0.19922$. You may also trade stock, whose Greeks are $(\Delta, \Gamma, \nu) = (1, 0, 0)$.

(a) Calculate the portfolio's net delta, gamma, and vega before adding a hedge. Include the sign of the short put.

(b) How many 100-strike calls should you buy or sell to make net gamma zero? What happens to net vega, to the precision of the table?

(c) What stock trade then makes net delta zero? State whether you buy or short the stock.

(d) Explain why one option trade was able to remove both gamma and vega here. Would the same conclusion necessarily hold if the available hedge option had a different expiry?

3 Delta-hedged P&L [35 points]

You buy one option at $S = \$100$ and immediately hedge its delta with stock. Over the next four trading days you rebalance the stock hedge after each day's move. For this exercise, take the option's Greeks as constant: $\Gamma = 0.040$ per dollar of stock movement, $\Theta = -\$0.032$ per trading day, and $\nu = \$0.20$ per one-percentage-point change in implied volatility. Use the Lecture 8 approximation, in dollars per one-share option,

$$\text{hedged P\&L} \;\approx\; \sum_{i=1}^{4} \left(\Theta + \tfrac{1}{2}\Gamma(\delta S_i)^2 \right) + \nu\, \delta\sigma_{\text{pp}},$$

where $\delta\sigma_{\text{pp}}$ is the total change in implied volatility measured in percentage points. Initially, the quote does not change. Consider two possible stock paths:

Path	Day 1	Day 2	Day 3	Day 4
Quiet	+\$1	−\$1	+\$1	−\$1
Active	+\$2	−\$2	+\$2	−\$2

(a) For each path, calculate separately the total theta loss, gamma gain, and approximate delta-hedged P&L. Both paths end at \$100. Why are their hedged results different?

(b) What root-mean-square daily stock move would make the gamma gain exactly offset the theta loss? Compare this break-even move with each path.

(c) Now suppose that, *only on the Active path*, implied volatility falls by two percentage points by the end of day 4. Approximate the additional vega P&L and the revised total. Explain how a position can earn from stock movement yet lose money overall.